Bäcklund transformation and nonlinear superposition formula of an extended Lotka - Volterra equation

1997 J. Phys. A: Math. Gen. 30 3635

(http://iopscience.iop.org/0305-4470/30/10/034)

J. Phys. A: Math. Gen. **30** (1997) 3635–3641. Printed in the UK PII: S0305-4470(97)79772-5

Bäcklund transformation and nonlinear superposition formula of an extended Lotka–Volterra equation

Xing-Biao Hu† and R K Bullough‡

† State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific Engineering Computing, Academia Sinica, PO Box 2719, Beijing 100080, People's Republic of China§

‡ Department of Mathematics, UMIST, PO Box 88, Manchester M60 1QD, UK

Received 21 November 1996, in final form 5 February 1997

Abstract. An extended Lotka–Volterra equation is considered, and a Bäcklund transformation and a nonlinear superposition formula given for it. Multiple soliton solutions are derived step by step as an application of the obtained results. Thus the N-soliton conjecture by Narita is confirmed.

1. Introduction

The so-called Lotka–Volterra equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}N_n = (N_{n-1} - N_{n+1})N_n \tag{1}$$

which expresses a sequence of ecological pre-predator processes [1]. It also finds applications in other areas such as plasma physics [2]. It is known that equation (1) is completely integrable, for example, (1) has an N-soliton solution [3] and infinite number of conserved quantities [1]. As an extended version of (1), Itoh [4] has proposed an equation

$$\frac{\mathrm{d}}{\mathrm{d}t}N_n = \sum_{r=1}^{k-1}(N_{n-r} - N_{n+r})N_n. \tag{2}$$

Much research on this equation has been conducted. Itoh [4] gave its conserved quantities with periodic boundary condition. Two soliton solutions for (2) on an infinite chain have been obtained by Narita [5]. Bogoyavlensky [6] has found the Lax form for (2). In [7], the recursion operator for (2) with $k = 3$ was given and higher symmetries were presented. It is noted that a higher-order version of (1) was also considered by Nagai and Satsuma [8].

In the following, we will follow the notations as in [5] and consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\prod_{i=0}^{m-1} a_{n-((m-1)/2)+i} = \prod_{i=0}^{k-1} a_{n+((m-1)/2)+i-(k-1)} - \prod_{i=0}^{k-1} a_{n-((m-1)/2)+i}$$
$$(m = 1, 2, \ldots; k = 1, 2, \ldots, m \neq k) \tag{3}$$

§ Mailing address.

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\prod_{i=0}^{m-1} a_{n-((m-1)/2)+i} = \left(\prod_{i=0}^{-k-1} a_{n+((m+1)/2)+i}\right)^{-1} - \left(\prod_{i=0}^{-k-1} a_{n-((m+1)/2)+i+k+1}\right)^{-1}$$
$$(m = 1, 2, \ldots; -k = 1, 2, \ldots). \qquad (4)$$

It is easily shown that if $m = 1$ in particular, (3) can be transformed into (2) on an infinite chain, using

$$N_n = \prod_{i=0}^{k-2} a_{n+i-(k/2)+1}.$$

By a transformation

$$a_n = \frac{f_{n-(k+1)/2} f_{n+(k+1)/2}}{f_{n-(k-1)/2} f_{n+(k-1)/2}}$$

equations (3) and (4) can be transformed into the following bilinear equation [5]:

$$[D_t \sinh(\tfrac{1}{2}mD_n) - 2\sinh(\tfrac{1}{2}kD_n)\sinh(\tfrac{1}{2}(m-k)D_n)] f_n \cdot f_n = 0. \qquad (5)$$

Equation (5) is reduced to the Lokta–Volterra equation or a differential-difference analogue of the KdV equation [9] corresponding to the choices of $2m = k$ or $m = -k$.

The purpose of this paper is to present a Bäcklund transformation and nonlinear superposition formula for (5). Using these results, multiple soliton solutions are derived step by step and therefore the N-soliton conjecture by Narita is confirmed.

This paper is organized as follows. In section 2 we give the Bäcklund transformation for the system equation (5). Then in section 3, we give a brief proof of a nonlinear superposition formula. Some particular solutions of equation (5) are then found through this formula. Finally, section 4 summarizes the results. A brief appendix lists some bilinear operator identities made use of in this paper.

2. A Bäcklund transformation for the extended Lotka–Volterra equation

In this section, we derive a Bäcklund transformation (BT) for equation (5). The result obtained is:

Proposition 1. A BT for (5) is

$$\exp(\tfrac{1}{2}(m-k)D_n) f_n \cdot f_n' = [\lambda \exp(\tfrac{1}{2}(m+k)D_n) + \mu \exp(\tfrac{1}{2}(k-m)D_n)] f_n \cdot f_n'$$
$$[D_t - \lambda \exp(kD_n) - \gamma] f_n \cdot f_n' = 0 \qquad (6)$$

where λ, μ and γ are arbitrary constants.

Proof. We only need to prove, by using (6), that

$$P \equiv [\exp(\tfrac{1}{2}mD_n) f_n' \cdot f_n'][D_t \sinh(\tfrac{1}{2}mD_n) - 2\sinh(\tfrac{1}{2}kD_n)\sinh(\tfrac{1}{2}(m-k)D_n)] f_n \cdot f_n$$
$$-[\exp(\tfrac{1}{2}mD_n) f_n \cdot f_n][D_t \sinh(\tfrac{1}{2}mD_n)$$
$$-2\sinh(\tfrac{1}{2}kD_n)\sinh(\tfrac{1}{2}(m-k)D_n)] f_n' \cdot f_n' = 0.$$

Making use of (A.1)–(A.4) and (6), we find P can be rewritten as

$$\begin{aligned}P &= 2\sinh(\tfrac{1}{2}mD_n)(D_t f_n\cdot f_n')\cdot f_n f_n' \\ &\quad -2\sinh(\tfrac{1}{2}kD_n)[\exp(\tfrac{1}{2}(m-k)D_n)f_n\cdot f_n'][\exp(\tfrac{1}{2}(k-m)D_n)f_n\cdot f_n'] \\ &= 2\sinh(\tfrac{1}{2}mD_n)(D_t f_n\cdot f_n')\cdot f_n f_n' \\ &\quad -2\lambda\sinh(\tfrac{1}{2}kD_n)[\exp(\tfrac{1}{2}(m+k)D_n)f_n\cdot f_n'][\exp(\tfrac{1}{2}(k-m)D_n)f_n\cdot f_n'] \\ &= 2\sinh(\tfrac{1}{2}mD_n)[(D_t-\lambda\exp(kD_n))f_n\cdot f_n']\cdot f_n f_n' = 0.\end{aligned}$$

Thus we have completed the proof of proposition 1. □

By use of (6), we can easily obtain the following solutions from the trivial solution $f_n = 1$:

$$f_n = 1 + \exp(2\eta)$$

with

$$\lambda = \frac{\sinh(p(m-k))}{\exp(pk)\sinh(pm)} \qquad \mu = 1-\lambda \qquad \gamma = -\lambda$$

and

$$f_n = n + \frac{k}{m}(m-k)t$$

with $\lambda = (m-k)/m$, $\mu = k/m$, $\gamma = (k-m)/m$; where

$$\eta = \omega t - pn + \eta^0 \qquad \omega = -\frac{\sinh(kp)\sinh(p(m-k))}{\sinh(mp)}$$

and p, η^0 are constants. In order to obtain more particular solutions, such as multi-soliton solutions, we need a nonlinear superposition formula for equation (5). This is found in the next section.

3. A nonlinear superposition formula

In the following, we shall simply denote, without confusion, $f_n(t) = f(n,t) = f(n)$ or f. The result reached is:

Proposition 2. Let f_0 be a solution of equation (5) and suppose that $f_i (i = 1, 2)$ are solutions of (5) which are related to f_0 under the BT equation (6) with parameters $(\lambda_i, \mu_i, \gamma_i)$, i.e. $f_0 \stackrel{(\lambda_i,\mu_i,\gamma_i)}{\rightarrow} f_i (i = 1, 2)$, $f_j \neq 0 (j = 0, 1, 2)$. Then f_{12} defined by

$$\exp(\tfrac{1}{2}kD_n)f_0\cdot f_{12} = c[\lambda_1\exp(-\tfrac{1}{2}kD_n) - \lambda_2\exp(\tfrac{1}{2}kD_n)]f_1\cdot f_2 \quad (c \text{ is a non-zero constant}) \tag{7}$$

is a new solution which is related to f_1 and f_2 under the BT (6) with parameters $(\lambda_2, \mu_2, \gamma_2)$ and $(\lambda_1, \mu_1, \gamma_1)$, respectively.

Proof. It suffices to show that

$$[\exp(\tfrac{1}{2}(m-k)D_n) - \lambda_2\exp(\tfrac{1}{2}(m+k)D_n) - \mu_2\exp(\tfrac{1}{2}(k-m)D_n)]f_1\cdot f_{12} = 0 \tag{8}$$

$$[\exp(\tfrac{1}{2}(m-k)D_n) - \lambda_1\exp(\tfrac{1}{2}(m+k)D_n) - \mu_1\exp(\tfrac{1}{2}(k-m)D_n)]f_2\cdot f_{12} = 0 \tag{9}$$

$$[D_t - \lambda_2\exp(kD_n) - \gamma_2]f_1\cdot f_{12} = 0 \tag{10}$$

$$[D_t - \lambda_1\exp(kD_n) - \gamma_1]f_2\cdot f_{12} = 0. \tag{11}$$

From

$$[\exp(\tfrac{1}{2}(m-k)D_n)f_0\cdot f_2][\exp(\tfrac{1}{2}(m-k)D_n)-\lambda_1\exp(\tfrac{1}{2}(m+k)D_n)$$
$$-\mu_1\exp(\tfrac{1}{2}(k-m)D_n)]f_0\cdot f_1-[\exp(\tfrac{1}{2}(m-k)D_n)f_0\cdot f_1][\exp(\tfrac{1}{2}(m-k)D_n)$$
$$-\lambda_2\exp(\tfrac{1}{2}(m+k)D_n)-\mu_2\exp(\tfrac{1}{2}(k-m)D_n)]f_0\cdot f_2=0$$

and

$$[\lambda_2\exp(\tfrac{1}{2}(m+k)D_n)f_0\cdot f_2][\exp(\tfrac{1}{2}(m-k)D_n)-\lambda_1\exp(\tfrac{1}{2}(m+k)D_n)$$
$$-\mu_1\exp(\tfrac{1}{2}(k-m)D_n)]f_0\cdot f_1$$
$$-[\lambda_1\exp(\tfrac{1}{2}(m+k)D_n)f_0\cdot f_1][\exp(\tfrac{1}{2}(m-k)D_n)$$
$$-\lambda_2\exp(\tfrac{1}{2}(m+k)D_n)-\mu_2\exp(\tfrac{1}{2}(k-m)D_n)]f_0\cdot f_2=0$$

we have, by use of equations (A.5), (7) and a detailed calculation,

$$\exp(\tfrac{1}{2}(k+m)D_n)f_0\cdot f_{12}=c[-\mu_1\exp(\tfrac{1}{2}(m-k)D_n)+\mu_2\exp(\tfrac{1}{2}(k-m)D_n)]f_1\cdot f_2. \quad (12)$$
$$\exp(\tfrac{1}{2}mD_n)f_0\cdot f_{12}=c[-\mu_1\lambda_2\exp(\tfrac{1}{2}mD_n)+\lambda_1\mu_2\exp(-\tfrac{1}{2}mD_n)]f_1\cdot f_2. \quad (13)$$

from which it follows that (8) and (9) hold by using (A.6). Furthermore, from

$$[(D_t-\lambda_1\exp(kD_n)-\gamma_1)f_0\cdot f_1]f_2-[(D_t-\lambda_2\exp(kD_n)-\gamma_2)f_0\cdot f_2]f_1=0$$

and

$$\lambda_2\left\{\exp\left(k\frac{\partial}{\partial n}\right)[D_t-\lambda_1\exp(kD_n)-\gamma_1]f_0\cdot f_1\right\}f_2(n)$$
$$-\lambda_1\left\{\exp\left(k\frac{\partial}{\partial n}\right)[D_t-\lambda_2\exp(kD_n)-\gamma_2]f_0\cdot f_2\right\}f_1(n)=0$$

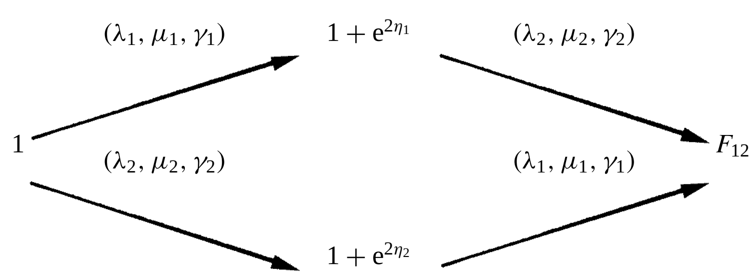
we have, by use of (A.7), (7) and a detailed calculation,

$$-D_tf_1(n)\cdot f_2(n)+(\gamma_2-\gamma_1)f_1(n)f_2(n)-\frac{1}{c}\exp(kD_n)f_0(n)\cdot f_{12}(n)=0 \quad (14)$$
$$\frac{1}{2k}D_tf_0(n+k)\cdot f_{12}(n)+\frac{1}{2}\lambda_2D_tf_1(n+k)\cdot f_2(n)+\frac{1}{2}\lambda_1D_tf_1(n)\cdot f_2(n+k)$$
$$+\lambda_2\gamma_1f_1(n+k)f_2(n)-\lambda_1\gamma_2f_1(n)f_2(n+k)=0. \quad (15)$$

Finally, we can show that (10) and (11) hold by using (7), (14) and (15). Thus we have completed the proof of proposition 2. □

As an application of this result, we give some particular solutions of (5).

Example 1. Choose $f_0=1$, $c=1/(\lambda_1-\lambda_2)$. It is easily verified that



where

$$F_{12} \equiv 1 + \frac{\lambda_1 - \lambda_2\,\mathrm{e}^{-2p_1k}}{\lambda_1 - \lambda_2}\,\mathrm{e}^{2\eta_1} + \frac{\lambda_2 - \lambda_1\,\mathrm{e}^{-2p_2k}}{\lambda_2 - \lambda_1}\,\mathrm{e}^{2\eta_2} + \frac{\lambda_1\,\mathrm{e}^{-2p_2k} - \lambda_2\,\mathrm{e}^{-2p_1k}}{\lambda_1 - \lambda_2}\,\mathrm{e}^{2(\eta_1+\eta_2)}$$

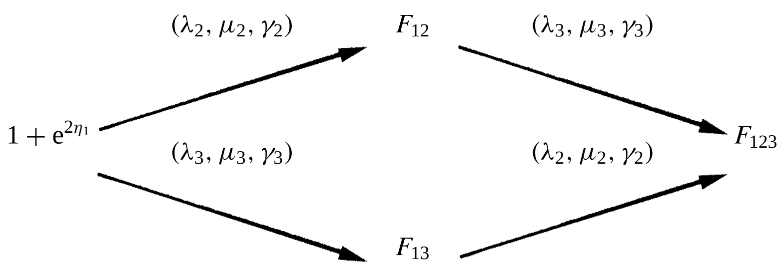
and

$$\eta_i = \omega_i t - p_i n + \eta_i^0 \qquad \omega_i = -\frac{\sinh(kp_i)\sinh(p_i(m-k))}{\sinh(mp_i)}$$

$$\lambda_i = \frac{\sinh(p_i(m-k))}{\exp(p_ik)\sinh(p_im)} \qquad \mu_i = 1 - \lambda_i \qquad \gamma_i = -\lambda_i \qquad (i = 1, 2).$$

p_i and η_i^0 are constants. Thus F_{12} is a two-soliton solution of (5).

Furthermore, we have



where

$$F_{13} \equiv 1 + \frac{\lambda_1 - \lambda_3\,\mathrm{e}^{-2p_1k}}{\lambda_1 - \lambda_3}\,\mathrm{e}^{2\eta_1} + \frac{\lambda_3 - \lambda_1\,\mathrm{e}^{-2p_3k}}{\lambda_3 - \lambda_1}\,\mathrm{e}^{2\eta_3} + \frac{\lambda_1\,\mathrm{e}^{-2p_3k} - \lambda_3\,\mathrm{e}^{-2p_1k}}{\lambda_1 - \lambda_3}\,\mathrm{e}^{2(\eta_1+\eta_3)}$$

and

$$F_{123} = 1 + K_1\,\mathrm{e}^{2\eta_1} + K_2\,\mathrm{e}^{2\eta_2} + K_3\,\mathrm{e}^{2\eta_3} + K_{12}\,\mathrm{e}^{2(\eta_1+\eta_2)} + K_{13}\,\mathrm{e}^{2(\eta_1+\eta_3)} + K_{23}\,\mathrm{e}^{2(\eta_2+\eta_3)} + K_{123}\,\mathrm{e}^{2(\eta_1+\eta_2+\eta_3)}$$

with

$$K_i = \prod_{\substack{j=1\\ j\neq i}}^{3} \frac{\lambda_i - \lambda_j\,\mathrm{e}^{-2p_ik}}{\lambda_i - \lambda_j}$$

$$K_{ij} = \frac{\lambda_i\,\mathrm{e}^{-2p_jk} - \lambda_j\,\mathrm{e}^{-2p_ik}}{\lambda_i - \lambda_j}\,\frac{\lambda_i - \lambda_{6-i-j}\,\mathrm{e}^{-2p_ik}}{\lambda_i - \lambda_{6-i-j}}\,\frac{\lambda_j - \lambda_{6-i-j}\,\mathrm{e}^{-2p_jk}}{\lambda_j - \lambda_{6-i-j}} \qquad i \neq j$$

$$K_{123} = \frac{\lambda_1\,\mathrm{e}^{-2p_2k} - \lambda_2\,\mathrm{e}^{-2p_1k}}{\lambda_1 - \lambda_2}\,\frac{\lambda_1\,\mathrm{e}^{-2p_3k} - \lambda_3\,\mathrm{e}^{-2p_1k}}{\lambda_1 - \lambda_3}\,\frac{\lambda_2\,\mathrm{e}^{-2p_3k} - \lambda_3\,\mathrm{e}^{-2p_2k}}{\lambda_2 - \lambda_3}.$$

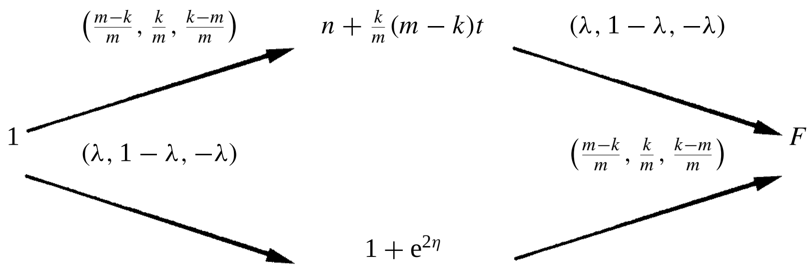
Then F_{123} is a three-soliton solution of (5) where

$$\eta_3 = \omega_3 t - p_3 n + \eta_3^0 \qquad \omega_3 = -\frac{\sinh(kp_3)\sinh(p_3(m-k))}{\sinh(mp_3)}$$

$$\lambda_3 = \frac{\sinh(p_3(m-k))}{\exp(p_3k)\sinh(p_3m)} \qquad \mu_3 = 1 - \lambda_3 \qquad \gamma_3 = -\lambda_3.$$

In general, we can obtain N-soliton solutions by successive use of (7).

Example 2. Choose $f_0 = 1, k = 1$, we have



where

$$F = -\lambda k + \left(\frac{m-k}{m} - \lambda\right)\left(n + \frac{k}{m}(m-k)t\right) - \lambda k\, e^{2\eta} + \left(\frac{m-k}{m} e^{-2kp} - \lambda\right)\left(n + \frac{k}{m}(m-k)t\right) e^{2\eta}$$

$$\eta = \omega t - pn + \eta^0 \qquad \omega = -\frac{\sinh(kp)\sinh(p(m-k))}{\sinh(mp)} \qquad \lambda = \frac{\sinh(p(m-k))}{\exp(pk)\sinh(pm)}.$$

and F is a solution of (5). Here p and η^0 are constants.

4. Summary

We have given a Backlund transformation for the extended Lotka–Volterra equation as well as a nonlinear superposition formula for it; and from the latter we find 2-, 3- and, in general, N-soliton solutions of the system and other solutions. Thus the N-soliton conjecture by Narita is confirmed.

Acknowledgments

This work was partially supported by the TWAS under research grant No 93-109 Maths/AS, the National Natural Science Foundation of China and the Chinese Academy of Sciences.

Appendix

The following bilinear operator identities hold for arbitrary functions a, b, c and d of n and t:

$$[D_t \sinh(\delta D_n)a \cdot a][\exp(\delta D_n)b \cdot b] - [D_t \sinh(\delta D_n)b \cdot b][\exp(\delta D_n)a \cdot a] = 2\sinh(\delta D_n)(D_t a \cdot b) \cdot ab \tag{A.1}$$

$$[\sinh(\delta_1 D_n)\sinh(\delta_2 D_n)a \cdot a][\exp((\delta_1+\delta_2)D_n)b \cdot b] - [\sinh(\delta_1 D_n)\sinh(\delta_2 D_n)b \cdot b][\exp((\delta_1+\delta_2)D_n)a \cdot a] = \sinh(\delta_1 D_n)[\exp(\delta_2 D_n)a \cdot b] \cdot [\exp(-\delta_2 D_n)a \cdot b] \tag{A.2}$$

$$\sinh(\delta D_n)a \cdot a = 0 \tag{A.3}$$

$$\sinh(\delta_1 D_n)[\exp((\delta_1+\delta_2)D_n)a \cdot b] \cdot [\exp((\delta_1-\delta_2)D_n)a \cdot b] = \sinh(\delta_2 D_n)[\exp(2\delta_1 D_n)a \cdot b] \cdot ab \tag{A.4}$$

$$[\exp(\delta_1 D_n)a \cdot b][\exp(\delta_2 D_n)a \cdot c] = \exp(\tfrac{1}{2}(\delta_1 + \delta_2)D_n)[\exp(\tfrac{1}{2}(\delta_1 - \delta_2)D_n)a \cdot a] \cdot [\exp(\tfrac{1}{2}(\delta_1 - \delta_2)D_n)c \cdot b] \quad \text{(A.5)}$$

$$[\exp(\delta_1 D_n)a \cdot b][\exp(\delta_2 D_n)c \cdot d] = \exp(\tfrac{1}{2}(\delta_1 - \delta_2)D_n)[\exp(\tfrac{1}{2}(\delta_1 + \delta_2)D_n)a \cdot d] \cdot [\exp(\tfrac{1}{2}(\delta_1 + \delta_2)D_n)c \cdot b] \quad \text{(A.6)}$$

$$(D_t a \cdot b)c - (D_t a \cdot c)b = -aD_t b \cdot c. \quad \text{(A.7)}$$

References

[1] Manakov S V 1974 *Sov. Phys.–JETP* **40** 269
[2] Zakharov V E, Musher S L and Rubenchik A M 1974 *JETP Lett.* **19** 151
[3] Hirota R and Satsuma J 1976 *J. Phys Soc. Japan* **40** 891
Hirota R and Satsuma J 1976 *Progr. Theor. Phys. Suppl.* **59** 64–100
[4] Itoh Y 1973 *Ann. Inst. Stat. Math.* **25** 635
Itoh Y 1987 *Prog. Theor. Phys.* **78** 507
[5] Narita K 1982 *J. Phys. Soc. Japan* **51** 1682
[6] Bogoyavlensky 1988 *Phys. Lett.* **134A** 34
[7] Zhang H-W, Tu G-Z, Oevel W and Fuchssteiner B 1991 *J. Math. Phys.* **32** 1908–18
[8] Nagai A and Satsuma J 1995 *J. Phys Soc. Japan* **64** 3669–74
[9] Hirota R 1977 *J. Phys Soc. Japan* **43** 1424–33